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Utility Representation of Homothetic Preferences*

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Abstract We offer an accessible proof of the existence of utility representation for continuous and homothetic preferences. When restricted to the Euclidean setting, our result is applicable to both positive and negative quantities while the standard proofs based on monotonicity are limited to the non-negative orthant. Our analysis also clarifies the roles played by key assumptions in the classic results. Our utility representation embodies an inherent asymmetry between regions of the alternatives space which may be interesting for extensions and applications.

Keywords Homotheticity, utility representation, monotonicity.

JEL Classification C60, D11.

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> The idea of a self-contained unity or limited whole is a fundamental instinctive concept. We see parts of things, we intuit whole things. We seem to know a great deal on the basis of very little.
>
> Iris Murdoch (1992) *Metaphysics as a Guide to Morals* (p.1)

1. INTRODUCTION

The classic Debreu's representation theorem (Debreu, 1954) ensures that a *complete*, *transitive* and *continuous* preference relation admits a utility function representation and that the representation is continuous. The result is well-known, but its proof is not elementary and does not offer easy intuition. More intuitive and accessible proofs are available if we add the assumption of *monotonicity* of preferences. (See, for example, Jehle and Reny, 2011, Theorem 1.1 and Mas-Colell, Whinston and Green, 1995, Proposition 3.C.1.[1])

This paper gives an alternative proof that replaces *monotonicity* with *homotheticity* and argues that our proof aids in understanding utility representation of preferences and may raise interesting possibilities for extensions & applications.

To put it briefly, homotheticity is scale-invariance; it says that preferences do not change as the alternatives are scaled up or down. Formally, a preference relation is homothetic if some alternatives x and y are such that x is preferred or indifferent to y, then αx is preferred or indifferent to αy where $\alpha > 0$ is an arbitrary positive number.

Why should we concern ourselves with homotheticity when it is obviously a very restrictive assumption? First, many popular textbook functional forms are based on homothetic preferences. Prominent examples include Cobb-Douglas and CES forms. It would be useful to have a firm grasp of the utility construction for these basic cases. Second, it is arguably reasonable over a sufficiently small domain. In fact, when experimental economists draw from "small-scale" experiments to make inferences about "real-world" behaviors of agents, they are implicitly drawing on homotheticity.

Third, the homotheticity assumption can play important roles in applications and extensions of microeconomic models, especially in terms of aggregation (Chipman, 1974; also see Hands, 2016, for historical and critical examination) so it is relevant to, e.g. representative agent models in macroeconomics. Constructing utility representations of homothetic preferences may throw light on such

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[1] In fact, the statement of Proposition 3.C.1 does not include monotonicity, but the proof adds the assumption as well as restricting to the nonnegative orthant of a finite-dimensional Euclidean space, ostensibly for convenience.

extensions. In fact, we shall argue that popular basic utility functions embody more symmetry than homotheticity imposes. Our construction using homotheticity allows an asymmetric utility representation between "gains" and "losses", which might be of interest in some contexts.

After setting down the preliminaries in Section 2, we state our main results in Section 3, with detailed proofs. Section 4 compares our proof with the standard proofs using monotonicity, with illustrative examples. We offer a very brief literature review in the final section as it involves technical discussions.

2. PRELIMINARIES

This section lays out the background, most of which is standard. The set of alternatives is denoted by X. Most accessible proofs (relying on monotonicity) restrict to the case $X = \mathbb{R}^n_+$, the nonnegative orthant of a finite-dimensional Euclidean space. But we shall first treat X as an abstract set and add further mathematical structures as needed. This can illuminate the role played by each mathematical structure in the proof. In the end, we shall state our results for a topological vector space as in Debreu (1954),[2] but put enough structure so that the proof will be elementary and accessible. We will also focus on $\mathbb{R}^n$ (rather than $\mathbb{R}^n_+$) in examples and discussions.

The agent has a binary preference relation $\succeq$ over X. (Note that $\succeq$ is not part of the description of X, but is part of the description of the agent.) We say that $\succeq$ is *complete* if for any $x, y \in X$, either $x \succeq y$ or $y \succeq x$ and *transitive*, if for any $x, y, z \in X$ such that $x \succeq y$ and $y \succeq z$, we have $x \succeq z$.[3] From any preference relation $\succeq$, we can define (i) the strict preference relation $\succ$ as $x \succ y$ whenever $y \not\succeq x$ and (ii) the indifference relation $\sim$ as $x \sim y$ whenever both $x \succeq y$ and $y \succeq x$.

We need a topological structure to define *continuity* of $\succeq$. We will freely use elementary notions and facts associated with a topological space such as open and closed sets, limit, convergence, and continuity of a function, as explained in e.g. Rudin, 1976, chapters 2 through 4.[4] We say that $\succeq$ is *continuous* if the sets $\{y \in X : y \succeq x\}$ and $\{y \in X : x \succeq y\}$ for any $x \in X$ are closed in the topology on

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[2] This, according to Debreu (1954), "involves no additional mathematical cost" but perhaps requires more mathematics than what a typical economist's toolbox contains.

[3] In the literature, a complete and transitive $\succeq$ is called a *weak order* or a *complete preorder* or a *rational preference relation*, depending on the authors.

[4] In brief, a topology on X is a collection of subsets ("open sets") of X such that (i) $\emptyset$ and X are open, (ii) any finite intersection of open sets is open and (iii) any arbitrary union of open sets is open. A complement of an open set is closed. We have a crucial result that a closed set contains all of its limit points, i.e. a convergent sequence of its elements has the limit in the set.

X. Debreu (1954) calls any such topology a *natural topology*. If X is given a natural topology for $\succeq$, then $\succeq$ is continuous by setup.

Debreu (1954) established that if $\succeq$ is complete, transitive, and continuous, then first, there exists a real-valued function $u : X \to \mathbb{R}$ such that $u(x) \geq u(y) \iff x \succeq y$ for any $x, y \in X$ and second, such u is continuous, i.e., if a sequence of $x_n \in X$ has the limit $\lim_{n\to\infty} x_n = x^* \in X$, then $\lim_{n\to\infty} u(x_n) = u(x^*)$.

Formally, Debreu (1954)'s main result is as follows.[5]

Theorem (Debreu, 1954, Theorem I)**.** *Let X be a separable[6] and connected[7] topological space with a natural topology for $\succeq$. If $\succeq$ is complete and transitive, then $\succeq$ has a continuous utility representation.*

The standard $\mathbb{R}^n_+$ is separable and connected with respect to the Euclidean topology. We can take the Euclidean topology to be a natural topology for $\succeq$. Hence the following textbook result follows.

Corollary. *Let $X = \mathbb{R}^n_+$. If $\succeq$ is complete, transitive, and continuous, then $\succeq$ has a continuous utility representation.*

Note that the entire $\mathbb{R}^n$ is also separable and connected, hence the Corollary can be extended to the entire finite-dimensional Euclidean space. However, the textbook proofs (based on monotonicity) are not applicable to $\mathbb{R}^n$, while our main result (based on homotheticity) will be.

3. HOMOTHETIC PREFERENCES AND UTILITY REPRESENTATION

3.1. HOMOTHETIC PREFERENCES

To define homotheticity of preferences, we need X to be embedded in some vector space. (Since we also need a topology on X for continuity assumption, the general setup is naturally a topological vector space.) The minimal structure we put on X for homotheticity is that it be a *cone*, i.e. if $x \in X$, then for any $\alpha > 0$, we have $\alpha x \in X$. We make a mathematically critical (see discussion in Section 5) but economically sensible assumption that $O \in X$, where O is the zero vector from the embedding vector space.

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[5] Debreu (1954) acknowledges that this result follows from Eilenberg (1941). In particular, (6.1) in Eilenberg (1941) can be seen as stating the same.

[6] X is separable if there is a countable subset $Z \subset X$ such that $\mathrm{cl}\, Z = X$ where $\mathrm{cl}\, Z$ is the closure of Z.

[7] X is connected if X cannot be partitioned into two nonempty, disjoint and closed sets.

In constructing utility representations, we will not use additions or convex combinations of vectors, so X need not be a linear subspace or a convex cone for our results, although such additional structures would be very useful if we were to utilize the utility representation in applications. Obviously, $\mathbb{R}^n_+$ and $\mathbb{R}^n$ do possess such extra structures.

Homotheticity of a preference relation requires preservation of preferences when vectors are scaled. There are two versions of the notion, by Katzner (1970) and Chipman (1974), respectively. Following Dow and Werlang (1992), we call Katzner's version *weak homotheticity* and show that the two are equivalent if continuity of preferences is assumed.

Definition 1 (Katzner). *A preference relation* $\succeq$ *is* weakly homothetic *if for* $x, y \in X$ *we have*

$$x \sim y \iff \alpha x \sim \alpha y, \quad \forall \alpha > 0.$$

Definition 2 (Chipman). *A preference relation* $\succeq$ *is* homothetic *if we have*

$$x \succeq y \iff \alpha x \succeq \alpha y, \quad \forall \alpha > 0.$$

It is immediate that a homothetic preference relation is weakly homothetic: if $x \sim y$, then both $x \succeq y$ and $y \succeq x$, which imply (by homotheticity) $\alpha x \succeq \alpha y$ and $\alpha y \succeq \alpha x$, so $\alpha x \sim \alpha y$.

The converse may not hold in general, but it does if we add continuity (and completeness) as stated by the following lemma. Dow and Werlang (1992) prove the same result using Debreu representation theorem, that is, assuming the existence of a continuous utility representation. In contrast, our proof does not presuppose the existence of a utility representation. More importantly, our proof employs a critical technique that will be used again in our main result.

Lemma 0. *If* $\succeq$ *is complete, continuous, and weakly homothetic, then it is homothetic.*

Proof of Lemma 0: Suppose on the contrary that for some $x \succ y$ there is $\alpha > 0$ such that $\alpha y \succeq \alpha x$. By weak homotheticity, $x \succ y$ implies $\alpha x \not\sim \alpha y$, so $\alpha y \succ \alpha x$. Let's assume for the moment that $\alpha > 1$.

Claim: There is β^* such that $\alpha > \beta^* > 1$ and $\beta^* x \sim \beta^* y$.

Proof of Claim (see Figure 1 for schematics of the proof): Consider the interval $[1, \alpha] \subset \mathbb{R}$. Let $A \equiv \{\beta \in [1, \alpha] : \beta x \succeq \beta y\}$ and $B \equiv \{\beta \in [1, \alpha] : \beta y \succeq \beta x\}$. A is non-empty since $1 \in A$. B is non-empty since $\alpha \in B$. Both A and B are closed in $\mathbb{R}$: to see this let $\{\beta_n\}$ be a sequence in A so that $\beta_n x \succeq \beta_n y$ for all n. If $\beta_n \to \beta_0$, then $\beta_0 x \succeq \beta_0 y$ by continuity of $\succeq$ and similarly for another

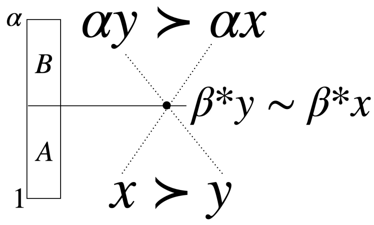

Figure 1: SCHEMATIC ILLUSTRATION OF CLAIM. The diagram schematically shows how Claim is proved.

sequence $\{\beta_n\}$ from B. By completeness, for any $\beta \in [1,\alpha]$, either $\beta x \succsim \beta y$ or $\beta y \succsim \beta x$, so $A \cup B = [1,\alpha]$. Since $[1,\alpha]$ is a connected real interval, $A \cap B$ must be non-empty. Let $\beta^* \in A \cap B$, then we have $\beta^* x \sim \beta^* y$. (End of Proof of Claim)

However, if $\beta^* x \sim \beta^* y$, then by weak homotheticity we must have $x \sim y$, a contradiction. Therefore, we conclude $x \succ y$ implies $\alpha x \succ \alpha y$.

If we reverse the momentary assumption so that $0 < \alpha < 1$, the Claim would change to $\alpha < \beta^* < 1$ and the argument will go through accordingly for the interval $[\alpha, 1]$. $\square$

Intuitively, the Claim says that if a preference reversal occurs across the two rays βx and βy for $\beta \in [1,\alpha]$, continuity and completeness force an indifference along the way. But this cannot happen if weak homotheticity is assumed. (Note that we use the fact that the closed real interval is connected.)

Remark 1. Some texts define homotheticity as a monotone transformation of homogeneous of degree 1 function (Varian, 1992, p.18 for production functions, p.146 for utility functions; de la Fuente, 2000, p.190; Jehle and Reny, 2011, p.612). But such a definition takes for granted the existence of numerical representation. We take the more primitive definition because our objective is to provide the representation.

Remark 2. (Mas-Colell, Whinston and Green, 1995, p.45) define homotheticity for "monotone" preference relations as preserving indifference when scaling up or down: $x \sim y \Rightarrow \alpha x \sim \alpha y$ for $\alpha \geq 0$. (also Carter, 2001, p.356.) This (*sans* monotonicity) corresponds to our weak homotheticity. It is unclear why Mas-Colell, Whinston and Green (1995) restrict to monotone preference relations in defining homotheticity. (Kreps, 2012, p.44) and (Rubinstein, 2024, p.56) define homotheticity as above.

3.2. UTILITY REPRESENTATION

We now state our main result. We have noted that a topological structure is required for using continuity and that a linear cone structure is needed for using homotheticity. We now assume X to be a cone embedded in a topological vector space and that $O \in X$. The standard Euclidean metric topology and linear structure should suffice for intuitions, but our proof will not invoke explicit topology or vector space structure.

Theorem 1 (existence of a utility representation)**.** *Let X be a cone within a topological vector space. Suppose $O \in X$. If a complete, transitive and continuous preference relation $\succeq$ defined on X is weakly homothetic, then there is a function $u : X \to \mathbb{R}$ that represents $\succeq$, i.e. $x \succeq y \iff u(x) \geq u(y)$ for $x, y \in X$.*

Remark. Compared to Debreu (1954), we do not explicitly assume X to be separable or connected while we add the assumption $O \in X$ (and homotheticity). See discussion in Section 5. Compared to textbook representation results (e.g., Mas-Colell, Whinston and Green, 1995, Proposition 3.C.1), we are not restricting X to $\mathbb{R}^n_+$ while replacing monotonicity with homotheticity. In fact, we will consider the entire $\mathbb{R}^n$ as our main application. See Section 4 for discussion and illustrations.

While homotheticity assumption is quite strong and the result itself is not new, novelty of this paper lies in the proof: how accessible it is and how it illuminates on previous textbook proofs based on monotonicity. For the sake of completeness, we also state the well-known characterization of the utility representation.

Theorem 2 (properties of the representation)**.** *The utility representation of Theorem 1 is continuous and homogeneous of degree 1, i.e., $u(\alpha x) = \alpha u(x)$ for $\alpha > 0$.*

Continuity is already ensured by Debreu's theorem, while homogeneity is added by homotheticity. Homogeneity introduces cardinality while Debreu's representation is ordinal.

3.2.1 Proof of Theorem 1

A scalar multiple of the zero vector is the zero vector, so homotheticity entails that for $\alpha > 0$, (i) $x \sim O \iff \alpha x \sim O$, (ii) $x \succ O \iff \alpha x \succ O$ and (iii) $O \succ x \iff O \succ \alpha x$. Hence, it is natural to classify vectors into three groups with respect to its relation to the zero vector. In other words, we partition the set X into three subsets: $X_0 = \{x \in X : x \sim O\}$, $\overline{X} = \{x \in X : x \succ O\}$, and $\underline{X} = \{x \in X : O \succ x\}$. By completeness, $X = X_0 \cup \overline{X} \cup \underline{X}$. The representation can proceed in the following steps.

Step 0: Set $u(O) = 0$. If $x \in X_0$, that is, if $x \sim O$, then $u(x) = 0$. By (weak) homotheticity, we have $u(\alpha x) = 0$ for all $\alpha > 0$ as well.

Step 1: Pick, if any, $\overline{x} \in \overline{X}$ so that $\overline{x} \succ O$. Set $u(\overline{x}) = 1$ and $u(\alpha\overline{x}) = \alpha$ for $\alpha > 0$.

Step 2: For any $x \in \overline{X}$, find $\alpha > 0$ such that $x \sim \alpha\overline{x}$. Set $u(x) = \alpha$.

Step 3: Pick, if any, $\underline{x} \in \underline{X}$ so that $O \succ \underline{x}$. Set $u(\underline{x}) = -1$ and $u(\alpha\underline{x}) = -\alpha$ for $\alpha > 0$.

Step 4: For any $x \in \underline{X}$, find $\alpha > 0$ such that $x \sim \alpha\underline{x}$. Set $u(x) = -\alpha$.

The five steps above complete the construction of $u(x)$ for all $x \in X$. If there is no $\overline{x}$ or $\underline{x}$, i.e. if either $\overline{X}$ or $\underline{X}$ is empty, the corresponding steps can be omitted.

We now prove, in a series of lemmas, that the steps are possible and indeed provide a utility representation of $\succeq$.

Lemma 1. *If $\overline{x} \succ O$, then we have $\alpha_1\overline{x} \succ \alpha_2\overline{x} \iff \alpha_1 > \alpha_2 (> 0)$.*

Proof of Lemma 1: Since $\overline{x} \succ O$, we have $\alpha\overline{x} \succ O$ for all $\alpha > 0$. Suppose $\alpha_1\overline{x} \succ \alpha_2\overline{x}$. It is obvious $\alpha_1 \neq \alpha_2$. Now suppose, on the contrary to the lemma, that $\alpha_1 < \alpha_2$. Let $\alpha_0 = \alpha_1/\alpha_2 < 1$. By homotheticity (apply $\alpha_2^{-1} > 0$ to both sides) $\alpha_1\overline{x} \succ \alpha_2\overline{x} \iff \alpha_0\overline{x} \succ \overline{x}$. Moreover, if we apply α_0 repeatedly, we have $\alpha_0^n\overline{x} \succ \alpha_0^{n-1}\overline{x} \succ \cdots \succ \alpha_0\overline{x} \succ \overline{x}$, which implies $\alpha_0^n\overline{x} \succ \overline{x}$ by transitivity. Since $0 < \alpha_0 < 1$, we have $\alpha_0^n \to 0$ as $n \to \infty$. By continuity, $\lim_{n\to\infty} \alpha_0^n\overline{x} = O \succeq \overline{x}$ (the limiting relation is weak because the set $\{y : O \succeq y\}$ is closed[8]). But this contradicts $\overline{x} \succ O$. Therefore, $\alpha_1 > \alpha_2$.

Now to take the converse, suppose $\alpha_1 > \alpha_2$. Suppose on the contrary that $\alpha_2\overline{x} \succeq \alpha_1\overline{x} \iff \alpha_0\overline{x} \succeq \overline{x}$, where $\alpha_0 = \alpha_2/\alpha_1 < 1$. Again apply α_0 repeatedly to obtain $\lim_{n\to\infty} \alpha_0^n\overline{x} = O \succeq \overline{x}$, a contradiction. Hence $\alpha_1\overline{x} \succ \alpha_2\overline{x}$. $\square$

Lemma 1 shows that Step 1 is valid, that is, preferences between the vectors of the form $\alpha\overline{x}$ can be represented by α as the utility value. The proof explicitly used all assumptions on $\succeq$ except for completeness, which was implicitly used via Lemma 0 (which makes weak homotheticity equivalent to homotheticity). Intuitively, monotonicity "emerges" along the ray $\alpha\overline{x}$ from the three key assumptions: homotheticity, continuity and $O \in X$. Among the vectors $\alpha\overline{x} \succ O$, those sufficiently close to O (with small α) are almost indifferent to O by continuity, which by homotheticity ensures that vectors farther from O (larger α) must be preferred to those closer ones.

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[8] I thank a referee for pointing this out.

Lemma 2. *For $x, y \in \overline{X}$ (i.e. $x \succ O$ and $y \succ O$), there is $\alpha > 0$ such that $x \sim \alpha y$.*

Proof of Lemma 2: Without loss of generality, let $x \succ y$.

By continuity, the set $\{y \in X : y \succ O\}$ is open. Then we can take a sufficiently small $0 < \alpha_0 < 1$ such that $y \succ \alpha_0 x \succ O$: If not, we must have $\alpha x \succeq y$ for all $\alpha > 0$. But then taking $\alpha \to 0$, we would obtain a contradiction $O \succeq y$.

Now consider the interval $[\alpha_0, 1] \subset \mathbb{R}_+$. Let $A \equiv \{\beta \in [\alpha_0, 1] : \beta y \succeq \alpha_0 x\}$ and $B \equiv \{\beta \in [\alpha_0, 1] : \alpha_0 x \succeq \beta y\}$. A is non-empty since $1 \in A$: by assumption we have $y \succ \alpha_0 x$. B is non-empty since $\alpha_0 \in B$: by assumption we have $x \succ y$ and by homotheticity $\alpha_0 x \succ \alpha_0 y$. By continuity, both A and B are closed in $\mathbb{R}$. By completeness, for any $\beta \in [\alpha_0, 1]$, either $\beta y \succeq \alpha_0 x$ or $\alpha_0 x \succeq \beta y$, so $A \cup B = [\alpha_0, 1]$, which is a connected real interval, hence $A \cap B$ must be non-empty. But if $\beta^* \in A \cap B$, then we have $\beta^* y \sim \alpha_0 x \iff (\beta^*/\alpha_0) y \sim x$ by homotheticity. Let $\alpha = \beta^*/\alpha_0$ and we have $x \sim \alpha y$. □

Lemma 2 shows that Step 2 is valid: for any $x \in \overline{X}$, we can find $\alpha > 0$ such that $x \sim \alpha \bar{x}$. Since $u(\alpha \bar{x}) = \alpha$, we also have $u(x) = \alpha$. The proof of Lemma 2 relies on the zero vector critically. First we use continuity and the zero vector to place y between x and $\alpha_0 x$ on a preference scale, then use the connectedness of a closed real interval, as in Claim within Lemma 0.

Steps 3 and 4 are analogous to Steps 1 and 2 with $\underline{x}$ taking the role of $\bar{x}$. The following lemmas correspond to Lemmas 1 and 2.

Lemma 3. *If $O \succ \underline{x}$, then we have $\alpha_1 \underline{x} \succ \alpha_2 \underline{x} \iff \alpha_1 < \alpha_2$*

Proof of Lemma 3: Suppose $\alpha_1 \underline{x} \succ \alpha_2 \underline{x}$ and $\alpha_1 > \alpha_2$. Then $\underline{x} \succ (\alpha_2/\alpha_1)\underline{x}$. Let $\alpha_0 = \alpha_2/\alpha_1 < 1$ and apply α_0 repeatedly to obtain $\underline{x} \succeq \lim_{n \to \infty} \alpha_0^n \underline{x} = O$, a contradiction. Similarly, suppose $\alpha_1 < \alpha_2$ and $\alpha_2 \underline{x} \succeq \alpha_1 \underline{x}$. Then let $\alpha_0 = \alpha_1/\alpha_2 < 1$ and apply α_0 repeatedly to obtain $\underline{x} \succeq O$, a contradiction. □

Lemma 3 shows that utility must be set inversely to α for $\alpha \underline{x} \in \underline{X}$. As $\alpha_1 < \alpha_2 \iff -\alpha_1 > -\alpha_2$, $u(\alpha \underline{x}) = -\alpha$ is one natural choice.

Lemma 4. *For $x, y \in \underline{X}$ (i.e. $0 \succ x$ and $0 \succ y$), there is $\alpha > 0$ such that $x \sim \alpha y$.*

Proof of Lemma 4 is essentially identical to that of Lemma 2 and is omitted. □

3.2.2 Proof of Theorem 2

The following two lemmas constitute Theorem 2.

Lemma 5 (homogeneity). *For the utility representation $u(\cdot)$ constructed by Theorem 1, we have $u(\lambda x) = \lambda u(x)$ for all $x \in X$ and $\lambda > 0$.*

Proof of Lemma 5: For any $x \in \overline{X}$, we find some α such that $x \sim \alpha\overline{x}$ and we have $u(x) = u(\alpha\overline{x}) = \alpha$. By (weak) homotheticity, $x \sim \alpha\overline{x} \iff \lambda x \sim \lambda\alpha\overline{x}$. Hence $u(\lambda x) = u(\lambda\alpha\overline{x}) = \lambda\alpha = \lambda u(x)$. For any $x \in \underline{X}$, we find α such that $x \sim \alpha\underline{x}$ and we have $u(x) = u(\alpha\underline{x}) = -\alpha$. By (weak) homotheticity, $x \sim \alpha\underline{x} \iff \lambda x \sim \lambda\alpha\underline{x}$. Hence $u(\lambda x) = u(\lambda\alpha\underline{x}) = -\lambda\alpha = \lambda u(x)$. For $x \in X_0$, $x \sim \lambda x \sim O$ for $\lambda > 0$ and $u(\lambda x) = 0 = \lambda u(x)$. □

Lemma 6 (continuity). *For the utility representation* $u(\cdot)$ *constructed by Theorem 1,* $\lim_{n\to\infty} u(x_n) = u(x^*)$ *for any sequence* $x_n \in X$ *such that* $\lim_{n\to\infty} x_n = x^*$.

Proof of Lemma 6: Consider $x \in \overline{X}$. Since $u(x) = \alpha$ when $x \sim \alpha\overline{x}$, we can write $x \sim u(x)\overline{x}$. Now consider a sequence of vectors x_n such that $x_n \to x^* \succ O$. By continuity, for sufficiently large n, we have $x_n \succ O$. So $x_n \sim u(x_n)\overline{x}$ and $x^* \sim u(x^*)\overline{x}$. As $x_n \to x^*$, we must have $u(x_n)\overline{x} \to u(x^*)\overline{x} \iff u(x_n) \to u(x^*)$. The reasoning is similar if $x^* \prec O$.

The case $x_n \to x^* \sim O$ is somewhat tricky,[9] because $X_0 = \{x : x \sim O\}$ is not open but closed. In other words we cannot say whether $x_n \succ O$ or $x_n \prec O$ or even $x_n \sim O$ for sufficiently large n. Since $u(x^*) = 0$, we must show $u(x_n) \to 0$ as $x_n \to x^*$. Since $x_n \to x^*$, we must be able to pick a subsequence y_m such that either $y_m \succ O$ or $y_m \prec O$ or $y_m \sim O$. If $y_m \succ O$, then $y_m \sim u(y_m)\overline{x} \to x^* \sim O \sim 0\overline{x}$ so that $u(y_m) \to 0$. If $y_m \prec O$, then $y_m \sim -u(y_m)\underline{x} \to x^* \sim O \sim 0\underline{x}$ so that $u(y_m) \to 0$. Finally if $y_m \sim O$, then $u(y_m) = 0$. □

4. MONOTONICITY VERSUS HOMOTHETICITY: ASYMMETRIC REPRESENTATION

In discussing aspects of our results, it would be convenient to focus on $\mathbb{R}^n$, particularly $\mathbb{R}^2$ for illustrations. Note that, first of all, while textbook discussion of utility representation is limited[10] to the nonnegative orthant $\mathbb{R}^n_+$, we consider the entire $\mathbb{R}^n$. A negative $x_i < 0$ may be interpreted either as supplying (instead of consuming) or as having a "loss" in the amount $|x_i|$.

To talk about monotone preferences, we need an order structure $\geq$ on X. It only needs to be a *partial order*, i.e. $\geq$ would be transitive but *not* complete. In addition, $\geq$ would be reflexive ($x \geq x$ for all x) and antisymmetric (if $x \geq y$ and $y \geq x$, then $x = y$). If $\geq$ were complete, then the utility representation would be trivial. If it were not reflexive or not antisymmetric, we would have unnatural instances (such as "$x > x$" or "$x \gneq y$ and $y \gneq x$").

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[9] I am grateful to a referee for pointing this out.

[10] (Kreps, 2012, Proposition 2.19 and Problem 2.12) presents utility representation for continuous and homothetic preferences but on $\mathbb{R}^n_+$.

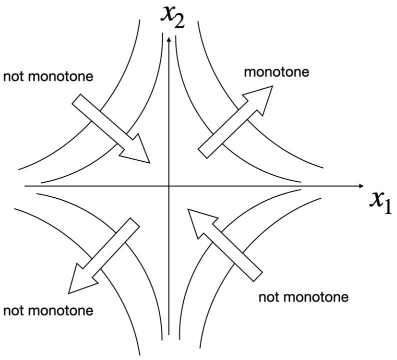
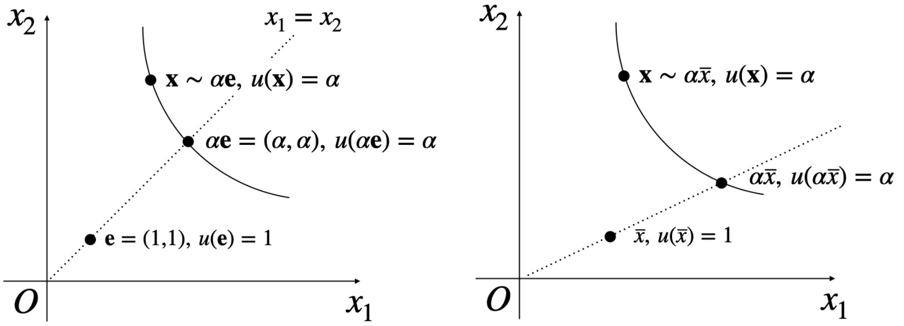


Figure 2: COBB-DOUGLAS INDIFFERENCE MAP. It is monotone only on the non-negative orthant (the first quadrant).

The vector inequality in $\mathbb{R}^n$ is a partial order: $(x_1,\ldots,x_n) \geq (y_1,\ldots,y_n) \iff x_i \geq y_i$ for all $i = 1,\ldots,n$. We can also define additional inequalities: (i) $x > y$ if $x \geq y$ but $y \not\geq x$ so that $x_i \geq y_i$ for all $i = 1,\ldots,n$ and $x_j > y_j$ for at least one $j = 1,\ldots,n$ and (ii) $x \gg y$ if $x_i > y_i$ for all $i = 1,\ldots,n$.

For use in a utility representation, we need a relatively weak notion of monotonicity: $x \succeq y$ whenever $x \geq y$ and $x \succ y$ if $x \gg y$. The familiar Cobb-Douglas utility $u(x_1,x_2) = x_1x_2$ satisfies such monotonicity on $\mathbb{R}^2_+$ but not on the entire $\mathbb{R}^2$: $u(2,2) > u(1,1)$ but $u(-1,-1) < u(-2,-2)$ although $(-1,-1) \gg (-2,-2)$. (See Figure 2.) In contrast, $u(x_1,x_2) = x_1x_2$ satisfies homotheticity on the entire $\mathbb{R}^2$ as can be easily checked. For $x = (x_1,x_2)$ and $y = (y_1,y_2)$ such that $x \succeq y \iff x_1x_2 \geq y_1y_2$, we have $u(\alpha x) = \alpha^2 x_1x_2 \geq \alpha^2 y_1y_2 = u(\alpha y)$ for any $\alpha > 0$.

The way homotheticity is used in our proof is similar to the way monotonicity is used in standard proofs, but homotheticity seems more versatile. The standard proofs pick a diagonal vector such as $\mathbf{e} = (1,1,\ldots,1) \in \mathbb{R}^n_+$ and use it as the "basis" to put utility values on all diagonal vectors $\alpha\mathbf{e} = (\alpha,\alpha,\ldots,\alpha)$, $\alpha > 0$. (See Figure 3(a).) By monotonicity $\alpha\mathbf{e} \succ \mathbf{e}$ if $\alpha > 1$ (and vice versa). So any increasing function $f : \mathbb{R}_+ \to \mathbb{R}$ can work as a representation: set $u(\alpha\mathbf{e}) = f(\alpha)$ with $f(1) = u(\mathbf{e})$. One natural choice is to set $u(\mathbf{e}) = 1$ and $u(\alpha\mathbf{e}) = \alpha$. (This is what (Mas-Colell, Whinston and Green, 1995, Proposition 3.C.1) do.) Such a choice may be said to be implicitly drawing on homotheticity, even though monotonicity notion itself has no connotation of scale-invariance. Finally, by continuity any non-diagonal vector $\mathbf{x}$ is "matched" to some diagonal vector $\alpha\mathbf{e}$ hence is given the utility $u(\mathbf{x}) = u(\alpha\mathbf{e}) = \alpha$.



(a) PROOF USING MONOTONICITY (b) PROOF USING HOMOTHETICITY

Figure 3: ILLUSTRATION OF PROOF STRATEGY. Two figures compare how representation is proved using monotonicity and homotheticity respectively.

Our proof strategy is almost the same, except that we are free to choose any vector $\bar{\mathbf{x}} \succ O$ and use $\bar{\mathbf{x}}$ as the basis to put utilities on the ray $\alpha\bar{\mathbf{x}}$, then appeal to continuity to put utility on arbitrary $\mathbf{x}$. (See Figure 3(b).) Moreover, homotheticity proof works beyond $\mathbb{R}^n_+$ because we also use another vector $\underline{x} \prec O$ as the basis as well. For concreteness, let us try constructing a utility representation for a Cobb-Douglas preference.

Example 1 (Cobb-Douglas)**.** We are given the preference relation underlying the Cobb-Douglas utility $\widehat{u}(x_1,x_2) = x_1x_2$ on $X = \mathbb{R}^2$. Then X_0 consists of two rays, i.e. the two orthogonal axes. $\overline{X}$ also consists of two pieces, the first and the third quadrant, while $\underline{X}$ consists of the second and the fourth quadrant. (See Figure 4.)

If we are to construct the utility function as outlined by the proof of Theorem 1, we first need to choose one $\bar{x} \in \overline{X}$ and one $\underline{x} \in \underline{X}$. (See Figure 5.) For example, let $\bar{x} = (2,1)$, then $u(2,1) = 1$ (although $\widehat{u}(2,1) = 2$). We then have $u(2\alpha,\alpha) = \alpha$ for any $\alpha > 0$ while $\widehat{u}(2\alpha,\alpha) = 2\alpha^2$. Now consider any vector $(x_1,x_2) \in \mathbb{R}^2_{++}$ such that $x_1 \neq 2x_2$. By continuity, we can find some $(2\beta,\beta) \sim (x_1,x_2)$. In fact, from $\widehat{u}$, we know $\widehat{u}(2\beta,\beta) = 2\beta^2$. So we can set $2\beta^2 = x_1x_2 \Rightarrow \beta = \sqrt{x_1x_2/2}$. Hence our construction yields $u(x_1,x_2) = \sqrt{x_1x_2/2}$.

For vectors $(x_1,x_2) \in \mathbb{R}^2_{--}$, as they are still in $\overline{X}$, we need to find some α so that $(x_1,x_2) \sim \alpha\bar{x} = (2\alpha,\alpha)$. Knowing the background $\widehat{u}$ we know that $u(-x_1,-x_2) = u(x_1,x_2)$, so $u(-2,-1) = u(2,1) = 1$. Therefore, the vector $(-2,-1)$ can work as if it is $\bar{x}$. And we still get $u(x_1,x_2) = \sqrt{x_1x_2/2}$ for both $(x_1,x_2) \gg O$ and $(x_1,x_2) \ll O$.

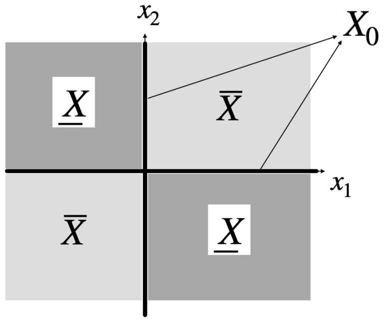


Figure 4: CLASSIFICATION OF X FOR COBB-DOUGLAS. For Cobb-Douglas, X_0 consists of two axes, while $\overline{X}$ and $\underline{X}$ each consists of two quadrants.

As for $\underline{X}$, we might choose $\underline{x} = (-3,1)$ and let $u(-3,1) = -1$. Note that $\bar{x}$ and $\underline{x}$ need not have any symmetry or relation between them. Then $u(-3\alpha,\alpha) = -\alpha$ for $\alpha > 0$, while $\widehat{u}(-3\alpha,\alpha) = -3\alpha^2$. For an arbitrary vector (x_1,x_2) with $x_1 < 0$ and $x_2 > 0$, we want to find $(-3\beta,\beta) \sim (x_1,x_2)$ so $-3\beta^2 = x_1x_2 < 0 \Rightarrow \beta = \sqrt{-x_1x_2/3}$. Hence $u(x_1,x_2) = -\sqrt{-x_1x_2/3}$. Again we can extend this representation to those vectors with $x_1 > 0$ and $x_2 < 0$. For example, $u(-1,1) = u(1,-1) = -\sqrt{1/3}$.

Therefore, we have shown that the preference relation underlying the Cobb-

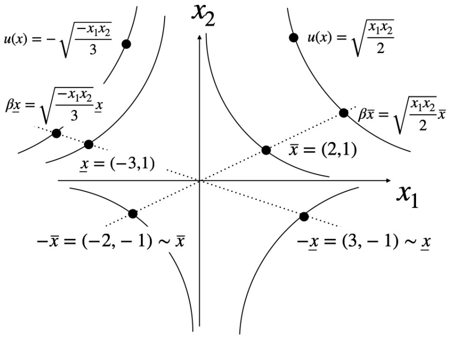


Figure 5: CONSTRUCTING A UTILITY REPRESENTATION FOR COBB-DOUGLAS. Utility representation involves choice of $\bar{x}$, $\underline{x}$ and finding relevant vectors on their rays.

Douglas utility function $\widehat{u}(x_1,x_2) = x_1x_2$ can also be represented by

$$u(x_1,x_2) = \begin{cases} \sqrt{\frac{x_1x_2}{2}}, & x_1x_2 > 0 \\ 0, & x_1x_2 = 0 \\ -\sqrt{\frac{-x_1x_2}{3}}, & x_1x_2 < 0 \end{cases} \sim \quad \widehat{u}(x_1,x_2) = x_1x_2.$$

While the background utility $\widehat{u}$ is homogeneous of degree 2, the constructed representation is homogeneous of degree 1 (as guaranteed by Theorem 2).

In contrast, the proof based on monotonicity cannot be applied here because $\widehat{u}(x_1,x_2) = x_1x_2$ is not monotone outside $\mathbb{R}^2_+$ (Figure 2). Hence the construction $u(\alpha\mathbf{e}) = \alpha$ doesn't work for $\alpha < 0$. We might observe that Cobb-Douglas preference has (reflexive) symmetry with respect to the origin (Figure 6(a)): $(-x_1,-x_2) \sim (x_1,x_2)$, which is an additional assumption[11] separate from monotonicity or homotheticity. This observation allows us to put utilities on vectors $(x_1,x_2) \in \mathbb{R}^2_{--}$. However, this diagonal-based approach does not work at all for the second and the fourth quadrant vectors, i.e. when $x_1x_2 < 0$.

The Cobb-Douglas preference embodies another ("rotational negative") symmetry so that $u(x_1,-x_2) = -u(x_1,x_2)$. (Figure 6(b)) Incorporating this as an additional assumption as well as the reflexive symmetry would yield a symmetric extension to the entire $\mathbb{R}^2$. But our homotheticity-based proof would work, in principle, without the additional symmetry assumptions. □

Note that in contrast to the standard Cobb-Douglas representation, our construction (*sans* rotational negative symmetry assumption) can give different "scal -es" to vectors with positive utility and vectors with negative utility. Once one fixes a formula for the utility function, it does possess a numerical scale with some cardinality which may be utilized in applications. (Cardinality is also evident from homogeneity of the utility representation.)

To emphasize versatility of the homotheticity-based approach, we present another example, this time with one good and one bad.

Example 2 (linear preference with one good, one bad)**.** Consider the preferences underlying the utility function $\widetilde{u}(x_1,x_2) = x_1 - x_2$. Good 1 is a "good" while good 2 is a "bad" and the slope of indifferences curves are constant (upward-sloping linear indifference map, Figure 7(a)).

Again this represents a homothetic (but not monotone) preference relation: $x_1 - x_2 \geq y_1 - y_2 \iff \alpha x_1 - \alpha x_2 \geq \alpha y_1 - \alpha y_2$ for $\alpha > 0$. X_0 consists of the

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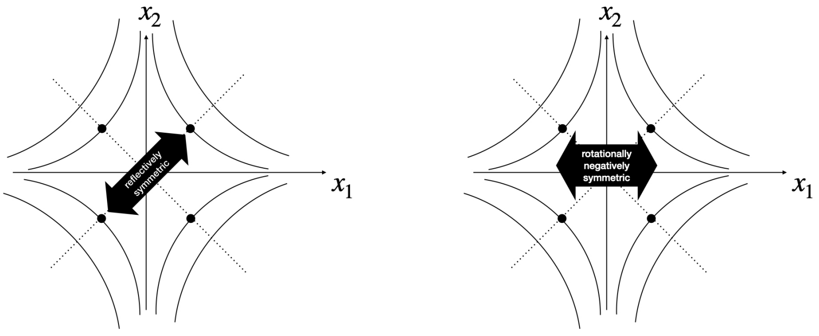
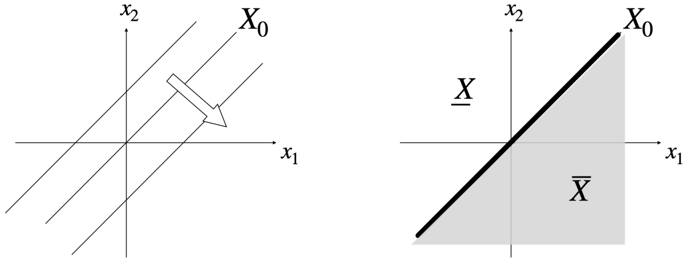
[11] This is a mathematical property of the Cobb-Douglas formula that exhibits a homothetic, but not monotone, preference relation and is not meant to represent any economically meaningful consumer model as it stands. I thank a referee for this point.



(a) REFLEXIVELY SYMMETRIC (b) ROTATIONALLY NEGATIVELY SYMMETRIC

Figure 6: SYMMETRY OF COBB-DOUGLAS PREFERENCE. Cobb-Douglas preference relation is symmetric.



(a) LINEAR INDIFFERENCE MAP (b) CLASSIFICATION OF X

Figure 7: LINEAR PREFERENCES WITH ONE GOOD, ONE BAD. The indifference curves straight lines and a line through the origin divides X into three regions.

45 degree line: $x_1 = x_2$. (See Figure 7(b).) We can pick $\bar{x}$ from its hypograph $x_2 < x_1$, say $\bar{x} = (3,1)$ with $u(3,1) = 1$ and $u(3\alpha,\alpha) = \alpha$. Then for any (x_1,x_2) such that $x_1 > x_2$, we find β such that $(3\beta,\beta) \sim (x_1,x_2) \iff 2\beta = x_1 - x_2$ so $u(x_1,x_2) = (x_1 - x_2)/2$. Similarly we pick $\underline{x}$ from the epigraph $x_1 > x_2$, say $\underline{x} = (1,4)$ with $u(1,4) = -1$ and $u(\alpha,4\alpha) = -\alpha$. Then for $x_1 < x_2$, $(\beta,4\beta) \sim (x_1,x_2) \iff -3\beta = x_1 - x_2$ so $u(x_1,x_2) = -\beta = (x_1 - x_2)/3$. So again, we built an asymmetric utility representation (Figure 8)

$$u(x_1,x_2) = \begin{cases} \frac{x_1-x_2}{2}, & x_1 > x_2 \\ 0, & x_1 = x_2 \\ \frac{x_1-x_2}{3}, & x_1 < x_2 \end{cases} \quad \sim \quad \widetilde{u}(x_1,x_2) = x_1 - x_2$$

The key aspect of our proof is that we pick two (arbitrary) vectors $\bar{x}(\succ O)$ and $\underline{x}(\prec O)$ as a set of basis. Even though we do not assume monotonicity (hence do not require order structure on X), homotheticity yields monotonicity among the vectors on the ray $\alpha\bar{x}$ ("goods"). In addition, homotheticity also yields negative monotonicity among the vectors on the ray $\alpha\underline{x}$ ("bads"). Then continuity allows us to link all vectors $\succ O$ with some $\alpha\bar{x}$ and all vectors $\prec O$ with some $\alpha\underline{x}$. All this procedure is preceded by first delineating the ray of vectors indifferent to O, so that the set of alternatives is classified into three regions. If the vectors indifferent to O all lie on a single ray, then this ray separates the rest of X into those preferred to O on one side and those less preferred to O on the other side as in Example 2. Even if X_0 consists of several distinct rays as in Example 1, the resulting partitions still represent either $\overline{X}$ or $\underline{X}$.

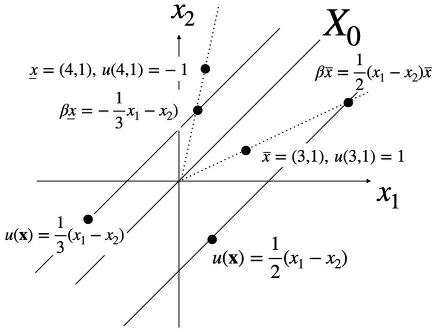


Figure 8: CONSTRUCTING A UTILITY REPRESENTATION FOR LINEAR PREFERENCE. Utility representation involves choice of $\bar{x}$, $\underline{x}$ and finding relevant vectors on their rays.

In particular, we do not impose symmetry between the choice of $\bar{x}$ and $\underline{x}$. If we replace $\underline{x}$ with some other $\underline{x}' \prec O$ while keeping $\bar{x} \succ O$, then the resulting utility representation will be partly identical (on those $\succ O$) and partly monotonically transformed (on those $\prec O$), while representing the same preferences. Perhaps this might be relevant for preference relations that treat gains and losses asymmetrically, as in prospect theory (Kahneman and Tversky, 1979).

In brief, there are four key ingredients in prospect theory: (i) reference dependence, (ii) loss aversion, (iii) diminishing sensitivity and (iv) probability weighting. (i), (ii), (iii) are captured by an S-shaped value function while (iv) is captured by an inverse-S-shaped probability weighting function. This paper has nothing to say about (iv), while homotheticity clearly rules out (iii).[12] However, our utility representation may accommodate (i) and (ii).

In other words, homotheticity-based construction, being homogeneous of degree 1, would yield an indirect utility (of money) with constant slope, contradicting the feature (iii) of prospect theory. But the indirect utility can have different slopes depending on the regions divided by the zero vector (reference point) hence accommodating (i) and (ii). Obviously this is only a sketch and we leave further development for future work.

5. A BRIEF LITERATURE REVIEW: TOPOLOGICAL CONSIDERATIONS INVOLVING CONTINUITY, SEPARABILITY, CONNECTEDNESS

The literature on utility representation per se is huge and beyond the scope of this paper. But there is a relatively small literature on utility representation of *homothetic* preferences. A major concern there has been topological aspects.

Throughout the paper as well as in most textbook treatments, continuity of preference relations is crucially used. Debreu (1954) presents the well known counterexample of a lexicographic preference relation on $\mathbb{R}^2$ to show it may be impossible to represent a discontinuous preference relation. A lexicographic preference relation, while complete and transitive, produces an indifference map where each vector is a stand-alone indifference "curve" (class). In a sense, there are too many different utility levels to be coherently assigned to $\mathbb{R}$. Mathematically $\mathbb{R}^2$ and $\mathbb{R}$ have the same cardinality of a continuum, so the two-dimensional vectors of real numbers *can* be linearly ordered, but it is impossible that such an ordering preserves the lexicographic preferences. (Kreps, 2012, Proposition 1.12) gives a general necessary and sufficient condition for utility representation

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[12] I thank a referee for pointing this out.

which does not assume continuity.[13] Dow and Werlang (1992) weaken the continuity assumption to upper semi-continuity. As our objective was not generality but accessibility of proofs, we have retained the continuity assumption.

Although Debreu's representation theorem is typically quoted for the case $X = \mathbb{R}^n_+$, Debreu (1954)'s argument makes it clear that he wants to present the result for more general settings. Hence his Theorem I, that we reproduced in Section 2, is for a separable and connected topological space X. Then he acknowledges that this result can be easily derived from Eilenberg (1941), so he makes a further generalization in Theorem II, which we shall not discuss in detail here,[14] except to note that the key of Theorem II is that it dispenses with the connectedness of X.

But we did not explicitly make the assumption that X is separable or connected. The reason Debreu (1954) assumes separability and connectedness is that he builds his proof from the case where the number of indifference curves is countable.[15] On the other hand, our proof strategy is not grounded in the countable case. Bosi, Candeal and Indurain (2000) show that the assumption $O \in X$ is critical in this regard, since without the assumption we would require separability (the existence of countable dense subset).

More specifically, Bosi, Candeal and Indurain (2000) establish a general representation result (Theorem 2) on a cone imbedded in a separable topological vector space and show that the result follows if O is in the cone (Proposition). But they seem to consider the assumption $O \in X$ restrictive, so their main result excludes the assumption at the expense of accessibility of the proof. In contrast, by introducing the assumption $O \in X$ explicitly, which we consider economically sensible, we were able to greatly simplify the proof.

Bosi (1998) is an earlier work that assumes all vectors are weakly preferred to the zero vector. That is, he excludes what we denoted as $\underline{X}$ from X so that $X = X_0 \cup \overline{X}$. Restricted to the Euclidean setting this amounts to taking $X = \mathbb{R}^n_+$.

As for connectedness, the reader may have noticed that we have utilized the connectedness, not of X, but of an interval on the real line in proofs of Claim in Lemma 0 and of Lemma 2 (and by extension Lemma 4). By the zero vector, the cone structure and continuity, we are able to connect various parts of X and to achieve the result. Note that, in the examples on Cobb-Douglas and linear preferences, the partitioning sets X_0, $\overline{X}$ and $\underline{X}$ are each connected.

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[13] But he notes that this result is "not very practical." (p.12)

[14] The proof was later corrected in Debreu (1964).

[15] Kreps (2012) gives a very accessible proof for utility representation over a countable X in Proposition 1.11.

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